Math 210A Lecture 22 Notes

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1 Properties of Linear Groups

1.1 The special linear group $\mathrm{SL}_n(F)$

Let $\mathbb{F}_q$ be the field with q elements, where q is a prime power. Later on, we will prove that a unique such field exists for each q.

Proposition 1.1. $\mathrm{SL}_n(F)$ *is generated by elementary matrices* $\{\{E_{i,j}(\alpha) : i \neq j, \alpha \in F\}$.

Proof. Let U be the unipotent group of upper triangular matrices with 1s as a diagonal. $U \trianglelefteq B$, the Borel subgroup of upper triangular matrices. U is nilpotent. $U^{\mathrm{ab}} \cong \mathbb{F}$, which is generated by the images of $E_{i,i+1}(\alpha)$. So U is generated by the elementary matrices.

$\mathrm{GL}_n(F) = BWB$, where $W = \iota(S_n)$, where $\iota : S_n \to \mathrm{GL}_n(F)$ sends σ to its permutation matrix. In fact, $\mathrm{GL}_n(F) = \coprod_{w \in W} BwB$, and $G = \mathrm{SL}_n(F) = \coprod_{w \in \iota(A_n)} B'wB'$, where $B' = B \cap G$. So $B \cong U \rtimes F^n$, where F^n is thought of as the diagonal matrices.

It suffices to show that the diagonal matrices or determinant 1 and permutation matrices of determinant 1 are in the subgroup generated by elementary matrices. For diagonal matrices, it suffices to show that we can get matrices of this form:

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & x & & & & \\ & & & \ddots & & & \\ & & & & x^{-1} & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

with only 2 non-identity entries. Note that

$$[E_{1,2}(\alpha), E_{2,1}(\alpha)] = \begin{bmatrix} 1+\alpha & \alpha \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\alpha & -\alpha \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+\alpha+\alpha^2 & -\alpha^2 \\ \alpha & 1-\alpha \end{bmatrix},$$

so

$$E_{1,2}\left(\frac{\alpha^2}{1-\alpha}\right) \cdot [E_{1,2}(\alpha), E_{2,1}(\alpha)] \cdot E_{2,1}\left(\frac{-\alpha}{1-\alpha}\right) = \begin{bmatrix} (1-\alpha)^{-1} & 0 \\ 0 & 1-\alpha \end{bmatrix}.$$

To get permutation matrices, we do something like this:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \qquad \square$$

Proposition 1.2. *The groups $\langle \{E_{i,j}(\alpha) : \alpha \in F\} \rangle$ are all conjugate.*

Proof. Let σ be an even permutation. Then $\iota(\sigma) E_{i,j} \iota(\sigma)^{-1} = E_{\sigma(i),\sigma(j)}(\alpha)$; this is just a change of basis. The rest is an exercise. $\square$

Proposition 1.3. $\mathrm{SL}_n(F) = [\mathrm{GL}_n F, \mathrm{GL}_n(F)]$ *unless $n = 2$ and $F \cong \mathbb{F}_2$ or $\mathbb{F}_3$.*

Proof. Note that $E_{i,j}(\alpha) = [E_{i,k}(\alpha).E_{k,j}(\alpha)]$ with $k \neq i, j$ for $n \geq 3$. For $n = 2$, we have

$$\left[\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} \alpha & \alpha\beta \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} \alpha^{-1} & -\alpha^{-1}\beta \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} 1 & (\alpha^2 - 1)\beta \\ 0 & 1 \end{bmatrix}.$$

We can choose $\beta \neq 0$ and $\alpha^2 \neq 1$ with $\alpha \neq 0$ iff $F \cong \mathbb{F}_2$ or $\mathbb{F}_3$. $\square$

Proposition 1.4. $\mathrm{SL}_n(F)$ *acts doubly transitively on the set of 1-dimensional subspaces of F^n.*

Proof. Given pairs of distinct nonzero vectors $(v_1, v_2), (w_1, w_2)$ with $Fv_1 \neq Fv_2$ and $Fw_1 \neq Fw_2$, there exists an $A \in \mathrm{GL}_n(F)$ such that $Av_i = w_i$ for $i = 1, 2$. Follow this by the matrix sending $w_1 \mapsto \det(A)^{-1} w_1$, $w_2 \mapsto w_2$, and all other basis elements to themselves. $\square$

1.2 The projective special linear group $\mathrm{PSL}_n(\mathbb{F}_q)$.

Theorem 1.1. *$PSL_n(\mathbb{F}_q)$ is simple for $n \geq 2$, unless $n = 2$ and $q \in \{2, 3\}$.*

Proof. Let P be the stabilizer of $\mathbb{F}_q e_1$ in $G = \mathrm{SL}_n(\mathbb{F}_q)$. These are matrices (with determinant 1) where the first column has zeros everywhere except the top left entry. P is maximal $< G$, and $P = \coprod_{w \in P \cap \iota(A_n)} B' w B'$. Consider the subgroup $K \trianglelefteq P$ of matrices with 1s on the diagonal and 0s above the diagonal except possibly for the first row.

Suppose $N \trianglelefteq G$. If $N \leq P$, then $N = gNg^{-1}$ stabilizes $g \cdot \mathbb{F}_q e_1$ for all $g \in G$. So N stabilizes $\mathbb{F}_q e_i$ for all i. Also, N stabilizes $\mathbb{F}_q(e_i + e_j)$ for all $i \neq j$. So $N \subseteq Z(\mathrm{SL}_n(\mathbb{F}_q))$.

If $N \not\leq P$, then $PN = G$, since G is maximal. Then $KN/N \trianglelefteq PN/N = G/N$, so $KN \trianglelefteq G$. We have that $E_{1,j}(\alpha) \in K$ for all $\alpha \in \mathbb{F}_q$ and $j \geq 2$. So since KN is normal, $E_{i,j}(\alpha) \in KN$ for all $i \neq j$ and $\alpha \in F$ by our second proposition. Then $G = KN$ by the first proposition. So $G/N \cong K/(K \cap N)$ is abelian. Then $N \geq G' = \mathrm{SL}_n(\mathbb{F}_q)$ by the third proposition. So $N = G$. $\square$